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NOTE

A Note on Weighted Polynomial Approximation with Varying Weights

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It is shown that if weighted polynomials $w^n P_n$ with $\deg P_n \leqslant n$ converge uniformly on the support of the extremal measure associated with w, then they converge to 0 everywhere else. It is also shown that uniform approximation on the support can always be characterized by a closed subset Z having the property that a function can be approximated if and only if it vanishes on Z. © 1996 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let Σ be a closed subset of the real line and let $w: \Sigma \to [0, \infty)$ be a weight on Σ which is admissible in the sense of Mhaskar and Saff [4]; see also [8]. That is,

(i) w is continuous;

(ii) the set $\{x \in \Sigma \mid w(x) > 0\}$ has positive logarithmic capacity;

(iii) if Σ is unbounded, then $|x| \, w(x) \to 0$ as $|x| \to \infty$, $x \in \Sigma$.

We use μ_w to denote the extremal measure associated with w. This measure is characterized by the fact that it minimizes the weighted energy integral

$$I_w(\mu) := \iint \log \frac{1}{|x-y| \, w(x) \, w(y)} \, d\mu(x) \, d\mu(y)$$

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among all probability measures μ on Σ. We use S_w to denote the support of μ_w. It is known that S_w is a compact set and w is strictly positive on S_w. For more information we refer the reader to [4–6, 8].

Mhaskar and Saff [4] proved that the sup norm of weighted polynomials $w^n P_n$, deg $P_n \leqslant n$, "lives" on S_w in the sense that if for a constant M

$$|w^n(x) P_n(x)| \leqslant M \qquad \text{for q.e.} \quad z \in S_w, \tag{1.1}$$

then

$$|w^n(x) P_n(x)| \leqslant M \qquad \text{for q.e.} \quad z \in \Sigma. \tag{1.2}$$

Here "for q.e." means for "quasi-every," i.e., except for a set of logarithmic capacity 0.

It is of considerable interest to determine the collection of functions that can be approximated by weighted polynomials $w^n P_n$, deg $P_n \leqslant n$; see [1, 2] and especially [8]. One of the results of [8] is that approximation is only interesting on S_w.

THEOREM 1 (Totik [8, Theorem 4.1]). *Assume Σ is a regular closed set and w is an admissible weight on Σ. Let $x_0 \in \Sigma \backslash S_w$. If a sequence $(w^n P_n)_n$,* deg $P_n \leqslant n$, *of weighted polynomials converges to a function f uniformly on $S_w \cup \{x_0\}$, then $f(x_0) = 0$.*

The proof of Theorem 1 is quite complicated and uses deep results from potential theory. Here we present a simpler proof based on the classical Stone–Weierstrass theorem, which at the same time proves a stronger result.

THEOREM 2. *Let Σ be a closed set such that for every neighborhood U of a point $x \in \Sigma$, the set $\Sigma \cap U$ has positive logarithmic capacity. Let w be an admissible weight on Σ. If a sequence $(w^n P_n)_n$,* deg $P_n \leqslant n$, *of weighted polynomials converges uniformly on S_w, then $w^n(x_0) P_n(x_0)$ tends to* 0 *for every $x_0 \in \Sigma \backslash S_w$.*

Note that the assumption about positive capacity is weaker than the assumption of regularity in Theorem 1. Also note that some assumption of this kind is necessary to exclude the situation where Σ consists of, say, an interval and an isolated point and $w \equiv 1$.

Our methods also prove the following theorem, which apparently has not been stated in this generality before.

THEOREM 3. *Let Σ be a closed set and w a continuous admissible weight on Σ. There exists a closed set $Z \subset S_w$, such that a continuous function f on S_w is the uniform limit of weighted polynomials $w^n P_n$ on S_w if and only if f vanishes on Z.*

It is possible that Z is empty, as the classical case $w \equiv 1$ on a compact set Σ shows.

The set Z depends of course on the behavior of the extremal measure μ_w. General results to determine Z were obtained by Totik [8] and the author [3], but a complete characterization of Z in terms of μ_w is not known.

2. PROOFS

For the proofs we need the Stone–Weierstrass approximation theorem in the following form. The collection of real-valued continuous functions on a compact space X is denoted by $C(X)$. If $\mathscr{A} \subset C(X)$ then $Z(\mathscr{A})$ denotes the set of $x \in X$ such that $f(x) = 0$ for every $f \in \mathscr{A}$.

THEOREM 4 (Stone–Weierstrass [7, Theorem 5]). *Let X be a compact Hausdorff space and $\mathscr{A} \subset C(X)$. Suppose $\mathscr{A}$ has the following four properties.*

(a) *If $f, g \in \mathscr{A}$, $\alpha, \beta \in \mathbf{R}$ then $\alpha f + \beta g \in \mathscr{A}$.*

(b) *If $f, g \in \mathscr{A}$ then $fg \in \mathscr{A}$.*

(c) *$\mathscr{A}$ is closed under uniform limits.*

(d) *If $x_1, x_2 \in X \backslash Z(\mathscr{A})$ then there is $f \in \mathscr{A}$ such that $f(x_1) \neq f(x_2)$.*

Then

$$\mathscr{A} = \{ f \in C(X) \mid f \equiv 0 \text{ on } Z(\mathscr{A}) \}.$$

We start with the proof of Theorem 3.

Proof of Theorem 3. Let $\mathscr{A}$ be the collection of all continuous functions f on S_w, such that $w^n P_n \to f$ uniformly on S_w, for some polynomials P_n, deg $P_n \leqslant n$.

It is obvious that $\mathscr{A}$ satisfies (a).

If $w^n P_n \to f$ and $w^n Q_n \to g$ uniformly on S_w, then with $R_{2n} := P_n Q_n$, $R_{2n+1} := P_{n+1} Q_n$, it is clear that $w^n R_n$ tends to fg uniformly on S_w. Therefore $\mathscr{A}$ satisfies (b).

From a straightforward diagonal argument it follows that (c) is satisfied.

To prove (d) we note that for $x_1 \in X \backslash Z(\mathscr{A})$ there is $g \in \mathscr{A}$ with $g(x_1) \neq 0$. Suppose $w^n P_n$ tends to g uniformly on S_w. Taking $Q_{n+1}(x) := (x - x_2) P_n(x)$ and $f(x) := w(x)(x - x_2) g(x)$, we find that $w^{n+1} Q_{n+1}$ converges uniformly to f, so that $f \in \mathscr{A}$. Since $f(x_1) \neq f(x_2)$, we see that (d) holds too.

Now Theorem 3 follows immediately from Theorem 4. ∎

Proof of Theorem 2. Suppose to the contrary that $w^n P_n$ converge to f_0 uniformly on S_w, but $w^n(x_0) P_n(x_0)$ does not tend to 0 for some $x_0 \in \Sigma \backslash S_w$.

Let $\mathscr{A}_0$ be the collection of all continuous functions f on S_w which are uniform limits of weighted polynomials $w^n Q_n$, $\deg Q_n \leqslant n$, with the additional property that $Q_n(x_0) = 0$. As in the proof of Theorem 3 it is easy to show that $\mathscr{A}_0$ satisfies the assumptions of Theorem 4. Therefore there is a set $Z_0 \subset S_w$ such that $f \in \mathscr{A}_0$ if and only if f vanishes on Z_0.

We claim that $f_0 \in \mathscr{A}_0$. Indeed, because $w^{n+1}(x)(x - x_0) P_n(x)$ tends to $w(x)(x - x_0) f_0(x)$ on S_w, we have that $w(x)(x - x_0) f_0(x)$ belongs to $\mathscr{A}_0$ and therefore that $w(x)(x - x_0) f_0(x)$ vanishes on Z_0. Since f_0 vanishes on S_w exactly where $w(x)(x - x_0) f_0(x)$ vanishes, it follows that $f_0 \in \mathscr{A}_0$.

Thus there are polynomials Q_n with $\deg Q_n \leqslant n$ and $Q_n(x_0) = 0$ such that $w^n Q_n$ tends to f_0 on S_w. Then $w^n(P_n - Q_n)$ converges to 0 uniformly on S_w. Because $w^n(x_0) P_n(x_0)$ does not tend to zero, it follows that there is an $\varepsilon > 0$ and an integer n such that $|w^n(P_n - Q_n)| \leqslant \varepsilon$ on S_w while $|w^n(P_n - Q_n)| > 2\varepsilon$ at x_0. By continuity this last inequality holds in the intersection of Σ with a full neighborhood of x_0. By assumption, the intersection has positive capacity and this gives a contradiction with (1.1), (1.2). Therefore the weighted polynomials $w^n P_n$ tend to 0 on $\Sigma \backslash S_w$. ∎

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